Continuity in nature and in mathematics: Du Châtelet and Boscovich

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Abstract

In the mid-eighteenth century, it was usually taken for granted that all curves described by a single mathematical function were continuous, which meant that they had a shape without bends and a well-defined derivative. In this paper I discuss arguments for this claim made by two authors, Emilie du Châtelet and Roger Boscovich. I show that according to them, the claim follows from the law of continuity, which also applies to natural processes, so that natural processes and mathematical functions have a shared characteristic of being continuous. However, there were certain problems with their argument, and they had to deal with a counterexample, namely a mathematical function that seemed to describe a discontinuous curve.

1. Introduction

The law of continuity was a central principle in eighteenth century physics, especially in the Leibnizian tradition. An interesting aspect of the law of continuity is that it played a role in different domains, in particular physics, metaphysics and mathematics. Seeing how it applies to these different domains can therefore learn us something about the relations between these domains, and in particular, about how mathematical continuity relates to continuity in nature. In the mid-eighteenth century, it was commonly thought that all mathematical functions that are expressed in a single equation describe a continuous curve (see Bottazzini 1986; Youschkevitch 1976). For a curve to be "continuous" meant that it was smooth with no cusps, having a well-defined tangent at each point; thus, this is close to our current conception of differentiability. The fact that mathematical functions have this characteristic was generally taken to follow from the law of continuity (Bottazzini 1986; Truesdell 1960). For Euler, it even held by definition: in a text from 1748, he defines a function as a combination of algebraic operations, and then writes: "One calls a curved line continuous when its nature is determined by one specific function of x" (quoted in Schubring 2005, 26).

In a well-known episode in the history of mathematics, Euler made discontinuous curves mathematically acceptable by using them in his treatment of the vibrating string, allowing for the shape of the string to be discontinuous (for example in the case of a plucked string which has a sharp angle). Euler´s acceptance of discontinuity in the case of the vibrating string has been described by Truesdell as "the greatest advance in scientific methodology in the entire century" (Truesdell 1960, 248; see also Bottazzini 1986, and Wilson 1991). It also led to controversy: d´Alembert and Lagrange protested that this left the derivative undefined at points where the string made a sharp angle, and that for this reason, the methods of differential calculus could not be applied. In 1759, Lagrange argued: "It seems unquestionable that the consequences which follow from the rules of the differential and integral calculus will always be illegitimate in all the cases where this law (of continuity) is not taken to occur" (quoted in Bottazzini 1986).

But while Euler gave mathematical treatment to strings that had discontinuous bends, he still defined discontinuous curves as curves that were described by different functions in different domains (e.g. a straight line connected to another straight line at an angle). Thus, even Euler held on to the idea that any curve that is described by a single mathematical function is continuous (Youschkevitch 1976). The idea that any curve that is described by a single mathematical function is continuous was thus widely accepted in the mid-eighteenth century, and it meant that at least as long as you could describe a system or process by means of a single function, there was no need to worry about the applicability of differential calculus. Yet, this idea turned out to be untenable later on; in the nineteenth century, mathematical functions were developed that were discontinuous and non-differentiable.

In this paper, I discuss two authors who explicitly defended the idea that all mathematical functions correspond to continuous curves, namely Emilie Du Châtelet and Roger Boscovich. They based this claim on the law of continuity. The law of continuity thus ensured differentiability; moreover, the law of continuity not only applied to mathematical functions but applied in a similar way to natural processes, so that there was a corresponding continuity in mathematics and in nature. However, their arguments for the claim were problematic, and they both struggled with a counterexample to the claim, namely a mathematical equation corresponding to a seemingly discontinuous curve.

2. The Law of Continuity and Continuous Functions

The law of continuity is mainly known through the work of Leibniz, which was also the main source of Du Châtelet´s and Boscovich´s accounts of the law of continuity. For Leibniz, this law played a role in his metaphysics and physics as well as mathematics (Schubring 2005). He gave different formulations of the law, and it is not always clear how these relate to each other, and whether they can all be reduced to the same basic principle.

One can make a distinction between three main versions of the law of continuity:

(1) Infinite divisibility: geometrical space and time are continuous in the sense of being infinitely divisible (in modern terms, they are dense).

(2) Continuous change: if a quantity changes from one value to another (e.g. a change from motion to rest), it goes through all the intermediate values; there can be no instantaneous jumps from one value to another.

(3) Inclusion of limits: in a sequence which ends in a limit, characteristics of the sequence also apply to the limit (Schubring 2005, 174). This principle plays a role in Leibniz´ differential calculus, as it implies that infinitesimals have the same properties

as ordinary numbers and may be treated as such. It also implies that rest can be considered as a type of motion, and a circle can be considered as a type of parabola.

The law of continuity can thus apply to space, time, numbers, curves, and physical quantities, among others. As regards curves, it is important to distinguish two senses in which curves can be continuous:

(A) They go through all intermediate values, so that there are no gaps in the curve (thus, the curve is subjected to (2) above). This corresponds to our present notion of continuity of functions.

(B) They have no cusps or sharp bends. This means that their tangent it subjected to (2): it changes by passing through all intermediate values. This roughly corresponds to our present notion of differentiability of functions.

3. Du Châtelet on the Law of Continuity

In her *Institutions de Physique* (1740), Emilie Du Châtelet introduces the law of continuity as follows:

> ...it is also to Mr. Leibniz that we owe the principle which is of great fruitfulness in physics, it is he who teaches us that nothing takes place in nature by jumps, and that a being never passes from one state to another without passing through all the states that we can conceive between them. (Du Châtelet 1740, 30).

Thus, her definition of the law of continuity corresponds to (2) above. She takes this to be a foundational principle in physics: she uses it to argue that there can be no hard bodies, and that the laws of motion must be such as to satisfy this principle (Du Châtelet 1740, 36-37). Although she attributes the principle to Leibniz, there are significant differences between Du Châtelet´s and Leibniz´ accounts of the law of continuity, and Du Châtelet´s account in fact shows more similarity to that of Johann Bernoulli (Bernoulli 1727; see Van Strien 2014; Heimann 1977). This holds in particular for the relation of the law of continuity to the principle of sufficient reason, for which Du Châtelet gives the following account:

> ...for each state in which a being finds itself, there must be a sufficient reason why this being finds itself in this state rather than in any other; and this reason can only be found in the antecedent state. This antecedent state thus contained something which has given rise to the current state which has followed it; so that these two states are connected in such a way that it is impossible to put another one in between; because if there were a possible state between the current state and that which immediately preceded it, nature would have left the first state without already being determined by the second to abandon the first; there would thus not have been a sufficient reason why it would have passed to this state rather than to any other possible state. Thus no being passes from one state to another, without

> passing through the intermediate states; just as one does not go from one city to another without going over the road between them. (Du Châtelet 1740, 30-31).[1]

Thus, a state is always a continuation of the previous state: it is causally determined by the previous state and only infinitesimally different from it. Du Châtelet interprets the principle of sufficient reason as stating that for everything that happens there must be a sufficient reason for it to come about, which can be located in the previous instant. The sufficient reason is thus an efficient cause. Moreover, Du Châtelet makes the requirement that it must be intelligible to us how the effect follows from the cause (Du Châtelet 1740, 26-27). If a state were very different from the state immediately preceding it, their causal link would be unintelligible. Therefore, all states must be a continuation of previous states, and all change has to be gradual.

After having discussed the law of continuity in the context of natural processes, Du Châtelet goes on to describe how the law applies to geometry:

> In geometry, where everything takes place with the greatest order, one sees that this rule is observed with an extreme exactness, for all changes that happens to lines which are one, that is to say a line that is itself, or lines which together form one and the same whole, all those changes, I say, are not completed until after the figure has passed through all the possible changes leading to the state it acquires: thus, a line that is concave towards an axis (...) does not all of a sudden become convex without passing through all the states that are in between concavity and convexity, and through all the degrees that can lead from one to the other... (Du Châtelet 1740, 31).

The law of continuity thus applies to curves which form a whole; this type of curves is contrasted with curves which are composed of different pieces, for example, half a circle connected to a straight line. She refers to such curves as "Figures batardes" and it is clear that she does not regard these as proper curves. Proper curves can be described by a single equation, or in Du Châtelet's words, they are "produced by the same law" (Du Châtelet 1740, 32). Apparently, the determination of such curves is subjected to the principle of sufficient reason. It follows that they cannot go from concave to convex without going through all intermediate degrees of bending; the curves thus don´t have cusps or sharp bends, which can be expressed in modern terms by saying that they have a continuous derivative.

Thus, there is continuity in nature as well as in geometry, and Du Châtelet remarks that "The same happens in nature as in geometry, and it was not without reason that Plato called the creator *the eternal geometer*" (Du Châtelet 1740, 33). However, a discussion of Zeno´s paradoxes led to problems with Du Châtelet´s account of continuity. In geometry, one can prove that a line is infinitely divisible; but Zeno's paradoxes led Du Châtelet to argue that there can be no such infinite divisibility in nature, and thus no continuity of the type (1) (Du Châtelet 1740, 183). It seems that this would undermine continuity of the type (2) as

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[1] This fragment was later copied in the entry on the law of continuity in the *Encyclopédie* of Diderot and D'Alembert, without mention of the source (in Van Strien (2014, I have discussed this fragment without attributing it to Du Châtelet).

well: how can there be continuous change when physical quantities are not infinitely divisible? However, a closer look at Du Châtelet´s definition of continuity reveals that it does not involve infinite divisibility. She defines the continuum by stating that the parts of a continuum are arranged in such away that one cannot place anything between two successive parts (Du Châtelet 1740, 101-102). And we have seen above that when describing continuous change from one state to another, she writes that for two states of which the one causes the other, "it is impossible to put another one in between". This is consistent with the idea that physical quantities take on discrete values, but that they change continuously in the sense that they pass through all these values one by one, without skipping any. A problem with this account of continuous change was pointed out by Maupertuis, who in (1750) wrote about the law of continuity:

> ...I do not even really know what this law is. When we suppose that the velocity increases or diminishes by degrees, wouldn't there still be passages from one degree to another? And doesn't the most imperceptible passage violate the law of continuity as much as the sudden destruction of the universe would? (Maupertuis 1750, 20).

If physical quantities take discrete values, then even a gradual change involves small discontinuous jumps from one value to the next. Thus, although Du Châtelet explicitly applied the law of continuity to nature as well as to geometry, establishing a shared characteristic of continuity in both, the fact that she ultimately argued that nature and geometry are fundamentally different with regard to divisibility led to problems with her account of continuous change.

4. Boscovich on the Law of Continuity

The law of continuity playes a central role in the theory that Boscovich is well-known for, namely his theory according to which matter consists of point particles which are unextended but are centers of force. The force between particles is repulsive at short distances, preventing the particles from touching each other, and at large distances it is attractive and equal to Newtonian gravity. In between, the force oscillates between attractive and repulsive a couple of times, and the specific form of the curve is supposed to explain different properties of matter such as elasticity and magnetism (on his theory, see Koznjak 2015, Guzzardi 2016).

Boscovich calls the law of continuity the "main basis" of this theory (Boscovich 1754/2002, 29) and claims that the entire theory can be derived from it (Boscovich 1754/2002, 213). He defines the law of continuity as the statement that there can be no change from one state to another without a transition through all intermediate states; his version of the law of continuity is thus also of type (2). Like Du Châtelet, he takes the law of continuity to exclude the possibility of hard bodies, since collisions between such bodies would involve a discontinuous change in their velocity. But Boscovich goes further. He argues that also collisions between elastic bodies would violate the law of continuity. If two elastic bodies collide, the velocity of their center of mass changes continuously, but points on their boundary surface touch each other in an instant at which their velocity changes instantaneously. Therefore, Boscovich concludes that there can be no collisions at all (Boscovich 1763/1922, 30). In this way, he arrives at a theory in which bodies have a strong repulsive force at short distances so that they never actually touch.

Boscovich works out the foundations for the law of continuity in his book *De Continuitatis Lege* (1754). He rejects the argument for the law of continuity given by Bernoulli and Du Châtelet, which is based on the principle of sufficient reason, because he rejects the principle of sufficient reason (see Koznjak, 2015). Instead he provides two other arguments for the law of continuity, which he calls the argument from induction and the argument from metaphysics. However, it turns out that both argument have their weaknesses.

According to the argument from induction, we see continuity everywhere in nature and in mathematics, and on this basis we are justified in inferring that there must always be continuity. Boscovich gives different examples of continuities we encounter, including planetary trajectories and sun rises, as well as geometric curves (Boscovich 1754/2002, 181-83). Now, a problem is that there are also cases in which we seem to observe discontinuity: very sudden processes such as explosions, and processes which include motion in a sharp angle, such as a reflection of a ray of light. Boscovich argues that such apparent counterexamples to continuity do not pose a problem as long as we can conceive a way in which they could be reconciled with continuity: explosions could be gradual and continuous processes that are just really quick, and light rays which appear to make a sharp angle may actually make a gradual bend (Boscovich 1763/1922, 119-21; 1754/2002, 207-9).

However, Boscovich's own theory of point particles leads him to the conclusion that certain apparently continuous processes are actually discontinuous. It follows from the theory of point particles that each body consists of a discrete number of particles. Because of this, when we draw a line representing the horizon, the particles making up the sun will actually rise above it one by one; thus, in this sense, sun rises are not continuous. Boscovich deals with this by restricting the law of continuity to the motion of fundamental particles (1763/1922, 145), and conceding that the continuity that we observe in natural processes is in some cases merely apparent.

But this makes the argument quite problematic: if observable processes that seem discontinuous can actually be continuous, and observable processes that seem continuous can actually be discontinuous, it is hard to see how there can be an inductive, empirical basis for the conclusion that processes on the fundamental level must always be continuous.

The second argument Boscovich offers for the law of continuity, the argument from metaphysics, is based on Aristotle´s account of the continuum: what characterizes the continuum is that successive parts have a common boundary. For a line, the boundary between two parts is a point, for a surface, the boundary between two parts is a line, etc. It follows that there can never be two points right next to each other without there being an interval between them. If you have a line segment which ends in a point, then taking away the endpoint leaves you with a line segment which does not end in a point and thus has no boundary. This, according to Boscovich, is impossible. In modern terms, he rejects the possibility of open intervals.

Now, the argument is as follows. Suppose that some variable, e. g. the position of a body, changes as a function of time. Suppose that at time *t*, there is a discontinuous jump in the value of the variable (see fig. 1).

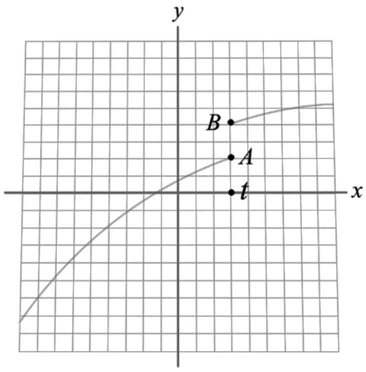

Fig. 1: Jump discontinuity

In the graph, each line representing a stretch of the trajectory has to end in a point, here denoted by A and B. Now, there are two possible cases:

- Either A and B correspond to the same instant *t*. Then at *t*, the variable has two different values.
- Or A and B do not correspond to the same instant *t*. Then, because time is mathematically continuous, there is a time interval in which the variable either has no value (if A comes before B) or two values (if B comes before A).

Thus, if there would be a jump discontinuity in the graph, this would mean that the depicted quantity either has two values or no value at a certain instant. Boscovich argues that this is impossible, both for physical and mathematical quantities. Physical quantities always have a single value at each moment in time; for example, it is metaphysically impossible that at a given time, a body has more or less than one position, or density, or temperature:

> …the distance of one body from another can never be altered suddenly, no more can its density; for there would be at one & the same time two distances, or two densities, a thing which is quite impossible without replication. Again, the change of heat, or cold, in thermometers, the change in the weight of the air in barometers, does not happen suddenly; for then there would necessarily be at one & the same time two different heights for the mercury in the instrument; & this could not possibly be the case. (Boscovich 1763/1922, 34)

Moreover, the argument also holds for mathematical quantities. Like Du Châtelet, Boscovich regards as proper curves those which can be expressed by a single equation (1754/2002, 195). For all such curves, it holds that each *x*-coordinate must correspond to a single *y*-coordinate (Boscovich, 1754/2002, 131-33), and for this reason there can be no jump discontinuities. In addition, with the same argument, he excludes jump discontinuities not only in the curve itself but also in its tangent: thus, there can be no curves with cusps (Boscovich, 1754/2002, 124-25).

The 'argument from metaphysics' thus applies to nature as well as mathematics, and shows that both processes in nature and geometrical curves described by a single equation have to be continuous. This argument depends essentially on the assumption that there can be no open intervals, while nowadays open intervals are regarded as completely unproblematic; with hindsight, we thus have to conclude that the argument does not work, and that Boscovich does not have a good argument for the law of continuity.

For Boscovich, the crucial assumption in the argument was that physical quantities and mathematical variables have a single value at every instant, and this is for him a metaphysical assumption. It is thus already assumed in the argument that the tangent of a curve is always well-defined and has exactly one value at each point, and that physical quantities such as position and temperature are similarly well-defined.

5. Apparent Counterexamples to the Continuity of Curves

In the previous two sections, we have seen how Du Châtelet and Boscovich attempted to demonstrate that any curve that can be described by a single equation has to be continuous and without cusps. Nowadays, we know of many examples of equations that describe curves that are not continuous. An extreme example are the Weierstrass functions, which, although being continuous in the modern sense, are not differentiable at any point and are thus not continuous in the eighteenth century sense; one may say that there is not a single point on the curve which is not a cusp. However, functions of this type were still unknown in the eighteenth century.

Another problematic case is the hyperbole ($y=1/x$) which has two branches going off to infinity so that there is a big jump in value between $x<0$ and $x>0$. This curve violates the continuity criterion according to which a small change in x has to correspond to a small change in y, and has no well-defined value or derivative at $x=0$. Because Euler was committed to the idea that any curve that is expressible by a single equation is continuous by definition, he argued that the curve of the hyperbola is continuous by definition, despite consisting of two separate branches (Youschkevitch, 1976, p. 33). Boscovich gives an argument for why hyperbolas don´t violate the law of continuity, based on his idea of 'transition through the infinite'. He argues that positive and negative infinity are connected: the number line can be seen as an infinite circle, so that if you move on the number axis in the direction of positive infinity, you can pass through infinity and arrive at the negative part of the axis (1754/2002, 89). He works out several examples of transitions through the infinite in geometry (see Guzzardi, 2016). Boscovich argues that in a hyperbola, when one of the branches goes of to positive infinity, it 'passes through the infinite' and appears again at negative infinity, so that there is a continuous connection.

Another class of problematic cases are curves with cusps such as the cycloid. The cycloid has a physical relevance, as it represents the path of a point of a rolling wheel and as such turns up in descriptions of nature. Another curve of this class is the curve described by *$x^2+y^3=0$,* which has a cusp in the origin (fig. 2). The latter type of curve was discussed explicitly by both Du Châtelet and Boscovich as a potential counterexample to their claim that curves described by a single equation are always continuous. Du Châtelet attempted to deal with this apparent counterexample by treating it as a limit of continuous curves – more precisely, a curve described by $x^2+y^3=0$ can be regarded as the limit of a curve of the type $x^2-y^2(a-y)=0$ where a goes to zero (fig. 3). The sharp bend should thus be regarded as an infinitely small loop, in which the tangent turns around, going through all intermediate values (Du Châtelet 1740, 32). A problem with this approach is that accepting the limit of a series of continuous functions or processes as continuous opens the door to accepting all kinds of discontinuities as continuous; furthermore, even if the point of the cusp is in fact an infinitely small node, its derivative is still not well defined in the origin.

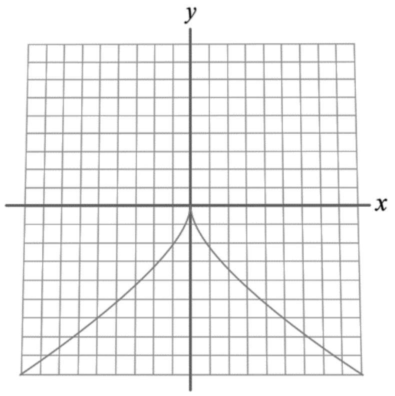

Fig. 2: Graph of $x^2 + y^3 = 0$

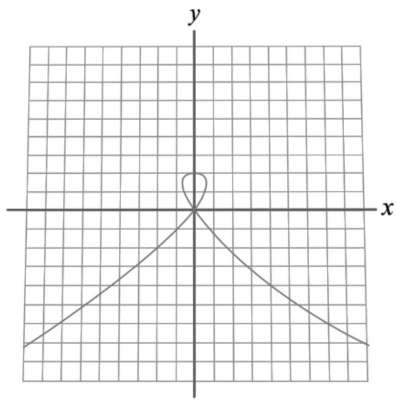

Fig. 3: Graph of $x^2 - y^2\left(\frac{1}{2} - y\right) = 0$

Boscovich argues that the peak of the curve is really a single point and not an infinitely small loop (1754/2002, 187), and puts forward his own argument for why this curve does not involve a violation of the law of continuity: the tangent of the curve makes a transition through the infinite. He argues that if you move gradually along the curve from left to right, the tangent changes continuously, going through all intermediate values, until at the top of the curve, the tangent is a vertical line. At this point, the derivative is infinite. Moving further along to the right, the tangent again changes continuously. The only thing that happens at the top is that there is a change in the direction of the tangent. At this point, the derivative, after going to infinity, passes "through the infinite". He argued that this solves the issue of the well-definedness of the derivative: the value of the derivative of the cusp is exactly the point in between positive and negative infinity.

Thus, like Du Châtelet, Boscovich argues that the curve is *actually* continuous, despite the fact that it seems to have a cusp. In contrast to Du Châtelet, Boscovich had an argument for the claim that the tangent of the curve is well-defined at each point; the argument, however, depends on his somewhat idiosyncratic ideas on transitions through the infinite.

6. Conclusions

We have seen how Du Châtelet and (in particular) Boscovich explicitly defended the idea that all mathematical functions correspond to continuous curves, and based this claim on the law of continuity. However, we have seen that their arguments were not without problems. In particular, they were both forced to argue that curves described by functions of the type $x^2+y^3=0$ are actually continuous, even though they clearly *look* discontinuous.

As mentioned in the introduction, the idea that all curves described by a single mathematical function are continuous ultimately proved to be untenable. One reason for this is that, as Youshkevitch (1976) has argued, the very distinction between curves which can be described by a single equation and curves which cannot be described in this manner has broken down. It was realized that sometimes one and the same curve can be either described by one equation or by two different equations, for example the curve described by $y = \sqrt{x^2}$ can also be described by $y = x$ for $x \geq 0$ and $y = -x$ for $x > 0$. Furthermore, Fourier analysis made it possible to give a single expression for all kinds of functions, so that they are continuous according to Euler´s definition. Thus, in the late eighteenth and early nineteenth century, the very distinction between continuous and discontinuous curves started to break down.

The claim that all mathematical functions correspond to continuous curves ensures differentiability and well-definedness of derivatives. Moreover, being a consequence of the law of continuity, which also applies to natural processes, it yielded a shared characteristic of continuity between natural processes and mathematical functions. However, the accounts of Du Châtelet and Boscovich were troubled by various problems and counterexamples. The idea that natural processes and mathematical functions share an aspect of continuity ultimately had to be given up.

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